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Generalized Subscalar Operators on Banach Spaces*

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INTRODUCTION

The concept of a subnormal operator, introduced by Halmos [4], has lead to the study of restrictions to invariant subspaces of many other kinds of operators. Saffern [14] studied subscalar operators on Hilbert spaces, C. Ionescu Tulcea on Banach spaces [5, 6], and Simpson [15] on locally convex spaces. Dowson [2] considered subspectral operators on Banach spaces.

The author has recently generalized the operators of Foias [3], Maeda [9-11], and Kantorovitz [8]. In this paper we will study restrictions of these operators to invariant subspaces. In Section 1 we generalize a theorem of C. Ionescu Tulcea [5] which theorem we then apply in Section 2 to characterize the generalized subscalar operators. In Section 3 we consider strong limits of generalized scalar operators and conditions under which every generalized subscalar operator is a generalized scalar operator.

We will use the terminology and notation of Bourbaki. The symbol N will denote the set $\{0, 1, 2,...\}$, N^* the set $\{1, 2, 3,...\}$, R the set of real numbers, R^2 or C the Euclidean (complex) plane, and T^1 the unit circle in R^2 considered as a one-dimensional manifold. For any topological space S, $\mathfrak{K}(S)$ will denote the set of all compact subsets of S; we also set $\mathfrak{K} = \mathfrak{K}(R^2)$. For $r \in N^*$, let $B(r) = \{z \in C : |z| \leqslant r\}$; note that $B(r) \in \mathfrak{K}$. If X is a locally compact space, $C^\infty(X)$ is the Banach algebra of bounded, complex-valued, continuous functions on X, $C_\infty(X)$ is the subalgebra of $C^\infty(X)$ consisting of those functions which vanish at infinity, and $\mathscr{K}(X)$ is the subalgebra of functions with compact support.

All vectors spaces will be over the complex field C. If E and F are vector spaces, each endowed with a topology, let $\mathscr{L}(E, F)$ be the set of continuous linear mappings of E into F. Whenever E and F are normed spaces, $\mathscr{L}(E, F)$ will be assumed to have the uniform (norm) topology. $\mathscr{L}(E)$ ($= \mathscr{L}(E, E)$) is a

* Some of the results of this paper were originally obtained, in less general form, by C. Ionescu Tulcea and the author [7].

Banach algebra with unit I. The image of $x \in E$ under $x' \in E'$ ($= \mathscr{L}(E, C)$) is denoted by $\langle x, x' \rangle$. The natural homomorphism $\nu : E \to E''$ ($= (E')'$) is defined by the equations $\langle x', \nu(x) \rangle = \langle x, x' \rangle$ for $x \in E$, $x' \in E'$.

Let E be a Banach space and $T \in \mathscr{L}(E)$. The *resolvent set*, $\rho(T)$, of T is the set of all $\lambda \in C$ such that $(\lambda - T)$ is invertible in $\mathscr{L}(E)$; the set $\mathrm{sp}(T) = \complement\rho(T)$ is called the *spectrum* of T. The *transpose* ${}^tT$ of T is the function $x' \to x' \cdot T$ of E' into E'; ${}^tT \in \mathscr{L}(E')$ and satisfies the equations $\langle x, {}^tTx' \rangle = \langle Tx, x' \rangle$ for $x \in E$, $x' \in E'$. One sees that $\mathrm{sp}(T) = \mathrm{sp}({}^tT)$. T is a *projection* if it is an idempotent in the algebra $\mathscr{L}(E)$.

Let E be a vector space. If p is a semi-norm on E, we will denote by E/p the factor space E/N where N is the nullspace of p. A filter base on a topological vector space will be called *bounded* if it contains a bounded set.

A *Michael algebra* [12] is a topological algebra $\mathscr{A}$ such that there exists fundamental system of neighborhoods of 0 in $\mathscr{A}$ consisting of convex, idempotent ($GG \subset G$) sets. Equivalently, $\mathscr{A}$ is a Michael algebra if it is a locally convex space and the set of continuous multiplicative ($p(xy) \leqslant p(x)\, p(y)$) semi-norms on $\mathscr{A}$ is cofinal in the set of all continuous semi-norms on $\mathscr{A}$. If $\mathscr{A}$ is an algebra and τ_0 is a topology on $\mathscr{A}$, let $m(\tau_0)$ be the supremum of topologies τ on $\mathscr{A}$ such that $\tau \subset \tau_0$ and $(\mathscr{A}, \tau)$ is a Michael algebra. Note that $(\mathscr{A}, m(\tau_0))$ is a Michael algebra.

For a nonempty open subset $\underset{\sim}{Q}$ of C the algebra $\mathscr{H}(\underset{\sim}{Q})$ of complex-valued functions holomorphic on $\underset{\sim}{Q}$ will be endowed with the topology of uniform convergence on compact subsets of $\underset{\sim}{Q}$. A *holomorphic function over* $K \in \mathfrak{K}$ is a function holomorphic on some neighborhood of K. Two holomorphic functions f and g over K are *equivalent* if $f \mid \underset{\sim}{Q} = g \mid \underset{\sim}{Q}$ for some neighborhood $\underset{\sim}{Q}$ of K. The set $\mathscr{H}(K)$ of equivalence classes of functions holomorphic over K is considered as an algebra in the natural way. When endowed with the "van Hove topology" (the inductive limit topology induced by the natural mappings of $\mathscr{H}(\underset{\sim}{Q})$ into $\mathscr{H}(K)$), $\mathscr{H}(K)$ is a topological algebra with unit 1. The symbol $\lambda(\lambda \in C)$ will denote the element $\lambda 1$ of $\mathscr{H}(K)$; the symbol z, the identity function of C onto itself considered as an element of $\mathscr{H}(K)$. Similarly, for $\lambda \in \complement K$, the function ψ_λ defined by the equation $\psi_\lambda(z) = 1/(\lambda - z)$ is the inverse of $(\lambda - z)$ in $\mathscr{H}(K)$. The basic properties of $\mathscr{H}(\underset{\sim}{Q})$ and $\mathscr{H}(K)$ are discussed in [18].

For $n \in N \cup \{\infty\}$ we will consider the algebra $C^n(R^2)$ [resp. $C^n(R)$] [resp. $C^n(\Delta)$ for Δ an interval in R] [resp. $C^n(T^1)$] of n-times continuously differentiable functions on R^2 [resp. R] [resp. Δ] [resp. T^1] with the topology of uniform convergence of a function and its derivatives on compact subsets of R^2 [resp. on compact subsets of R] [resp. on compact subsets of Δ] [resp. on T^1]. All of these algebras are Michael algebras.

1. The Dilation Theorems

In this section we prove two general theorems asserting the existence of dilations of families of operators on a normed space. The second theorem will be used to characterize generalized subscalar operators in Section 2.

Before stating the first theorem we introduce the notations which will be used in the proof. Let $\mathscr{A}$ be an algebra, E a vector space, and U a linear mapping of $\mathscr{A}$ into $\mathscr{L}^*(E)$. Define functions ω_a (for each $a \in \mathscr{A}$) and ω from the tensor product $\mathscr{A} \otimes E$ into E by

$$\omega_a \left(\sum b_j \otimes y_j\right) = \sum U(ab_j)\, y_j\,, \qquad \omega \left(\sum b_j \otimes y_j\right) = \sum U(b_j)\, y_j\,.$$

These functions are well-defined since $(b, y) \to U(b)\, y$ and $(b, y) \to U(ab)\, y$ ($a \in \mathscr{A}$) are bilinear mappings of $\mathscr{A} \times E$ into E. Note that if $\mathscr{A}$ has a unit 1, then $\omega = \omega_1$. If N is a subspace of $\mathscr{A} \otimes E$, $a \in \mathscr{A}$, and $x \in E$, then the symbol $a \dot{\otimes} x$ will denote the coset of $a \otimes x$ in $(\mathscr{A} \otimes E)/N$.

Definition. Let E be a normed space, $\mathscr{A}$ an algebra endowed with a topology, and U a continuous linear mapping of $\mathscr{A}$ into $\mathscr{L}(E)$. An object $(\tilde{E}, i, P, V)$ is called a *dilation* of U if $\tilde{E}$ is a Banach space, i is a linear bicontinuous injection of E into $\tilde{E}$, P is a bounded projection of $\tilde{E}$ onto the closure of $i(E)$, and V is a continuous representation of $\mathscr{A}$ into $\mathscr{L}(\tilde{E})$ such that $PV(a)\, i = iU(a)$ for every $a \in \mathscr{A}$. A dilation $(\tilde{E}, i, P, V)$ is called *minimal* if $\{V(a)\, ix : a \in \mathscr{A}, x \in E\}$ is total in $\tilde{E}$.

Theorem 1.1. *Let E be a normed space, $\mathscr{A}$ a normed algebra, and U a continuous linear mapping of $\mathscr{A}$ into $\mathscr{L}(E)$. Let $(u_\lambda)_{\lambda\in\Lambda}$ be a bounded net in $\mathscr{A}$ such that, for any $x \in E$, $b \in \mathscr{A}$,*

$$\lim_{\lambda\in\Lambda} \sup_{\|a\|\leqslant 1} \| U(au_\lambda - a)\, x \| = \lim_{\lambda\in\Lambda} \sup_{\|a\|\leqslant 1} \| U(au_\lambda b - ab)\, x \| = 0,$$

$$\lim_{\lambda\in\Lambda} U(u_\lambda)\, x = x, \qquad \textit{and} \qquad \lim_{\lambda\in\Lambda} U(u_\lambda b)\, x = U(b)\, x.$$

Then there exists a minimal dilation $(\tilde{E}, i, P, V)$ of U having the following properties:

(1.1.1) $\lim_{\lambda\in\Lambda} V(u_\lambda) = I$ *strongly in* $\mathscr{L}(\tilde{E})$;

(1.1.2) *for* $b \in \mathscr{A}$, $V(b)\, i = iU(b)$ *if and only if* $U(ab) = U(a)\, U(b)$ *for every* $a \in \mathscr{A}$;

(1.1.3) *for* $b \in \mathscr{A}$, $V(b) = 0$ *if and only if* $U(abc) = 0$ *for every* $a \in \mathscr{A}$, $c \in \mathscr{A}$;

(1.1.4) *if $\mathscr{F}$ is a bounded filter base on $\mathscr{A}$ such that*

$$\lim_{b,\mathscr{F}} \sup_{\|a\|\leqslant 1, \|c\|\leqslant 1} \| U(abc)\, x \| = 0$$

for every $x \in E$, then $\lim_{\mathscr{F}} V = 0$ strongly in $\mathscr{L}(E)$;

(1.1.5) $\| V \| \leqslant 1$, $\| P \| \leqslant \sigma^2 \| U \|$ *and* $x \leqslant \sigma \| i(x) \| \leqslant \sigma^2 \| U \| \| x \|$ *for every $x \in E$, where $\sigma = \sup_{\lambda \in \Lambda} \| u_\lambda \|$.*

Proof. Define a semi-norm p on $\mathscr{A} \otimes E$ by

$$p(t) = \sup_{\|a\|\leqslant 1} \| \omega_a(t) \|,$$

and let $\tilde{E}$ be the completion of $(\mathscr{A} \otimes E)/p$. For every $x \in E$, the net $(u_\lambda \dot{\otimes} x)_{\lambda\in\Lambda}$ is Cauchy in $\tilde{E}$ since

$$\| u_\lambda \dot{\otimes} x - u_{\lambda'} \dot{\otimes} x \| \leqslant \sup_{\|a\|\leqslant 1} \| U(au_\lambda - a)\, x \| + \sup_{\|a\|\leqslant 1} \| U(au_{\lambda'} - a)\, x \|.$$

Therefore we can define $i(x) = \lim_{\lambda\in\Lambda} u_\lambda \dot{\otimes} x$ for $x \in E$.

Before defining P and V we first establish several properties of $\tilde{E}$ and i. For any $x \in E$, $b \in \mathscr{A}$, $t = \sum b_j \otimes y_j \in \mathscr{A} \otimes E$,

(1) $\| b \dot{\otimes} x \| \leqslant \sigma \sup_{\|a\|\leqslant 1, \|c\|\leqslant 1} \| U(abc)\, x \| \leqslant \sigma \| U \| \| b \| \| x \|$,

(2) $\| x \| \leqslant \sigma \| i(x) \| \leqslant \sigma^2 \| U \| \| x \|$,

(3) $\| \omega(t) \| \leqslant \sigma p(t)$, $\| \omega_b(t) \| \leqslant \| b \| p(t)$,

(4) $p(\sum bb_j \otimes y_j) \leqslant \| b \| p(t)$,

(5) $\| i\omega(t) \| \leqslant \sigma^2 \| U \| p(t)$.

Proof of (1):

$$\begin{aligned} \| b \dot{\otimes} x \| &= \sup_{\|a\|\leqslant 1} \| U(ab)\, x \| \\ &= \sup_{\|a\|\leqslant 1} \lim_{\lambda\in\Lambda} \| U(abu_\lambda)\, x \| \leqslant \sup_{\|a\|\leqslant 1, \lambda\in\Lambda} \| U(abu_\lambda)\, x \| \\ &\leqslant \sigma \sup_{\|a\|\leqslant 1, \|c\|\leqslant 1} \| U(abc)\, x \|. \end{aligned}$$

The second inequality is obvious.

Proof of (2):

$$\begin{aligned} \sigma \| i(x) \| &= \sigma \lim_{\lambda\in\Lambda} \sup_{\|a\|\leqslant 1} \| U(au_\lambda)\, x \| \geqslant \sigma \sup_{\|a\|\leqslant 1} \lim_{\lambda\in\Lambda} \| U(au_\lambda)\, x \| \\ &= \sigma \sup_{\|a\|\leqslant 1} \| U(a)\, x \| \geqslant \sup_{\lambda\in\Lambda} \| U(u_\lambda)\, x \| \\ &\geqslant \lim_{\lambda\in\Lambda} \| U(u_\lambda)\, x \| = \| x \|. \end{aligned}$$

The second inequality is obvious.

PROOF OF (3): The fact that $\| \omega_b(t) \| \leqslant \| b \| p(t)$ is obvious.

$$\| \omega(t) \| = \lim_{\lambda \in \Lambda} \| \omega_{u_\lambda}(t) \| \leqslant \sup_{\lambda \in \Lambda} \| u_\lambda \| p(t) = \sigma p(t).$$

PROOF OF (4): Using (3),

$$p\left(\sum bb_j \otimes y_j\right) = \sup_{\|a\| \leqslant 1} \| \omega_{ab}(t) \| \leqslant \sup_{\|a\| \leqslant 1} \| ab \| p(t) \leqslant \| b \| p(t).$$

Now (5) follows easily from (2) and (3).

By (3) ω [resp. $\omega_b(b \in \mathscr{A})$] induces a linear continuous mapping $\tilde{\omega}$ [resp. $\tilde{\omega}_b$] from $(\mathscr{A} \otimes E)/p$ into E such that $\tilde{\omega}(a \dot{\otimes} x) = \omega(a \otimes x)$ [resp. $\tilde{\omega}_b(a \dot{\otimes} x) = \omega_b(a \otimes x)$] for each $a \in \mathscr{A}$, $x \in E$. Since $i\tilde{\omega} \in \mathscr{L}((\mathscr{A} \otimes E)/p)$, $i\tilde{\omega}$ can be extended to $P \in \mathscr{L}(\tilde{E})$ such that

$$\| P \| = \| i\tilde{\omega} \| \leqslant \| i \| \| \tilde{\omega} \| \leqslant \sigma^2 \| U \|$$

by (5). For $\sum b_j \otimes y_j \in \mathscr{A} \otimes E$, $b \in \mathscr{A}$ define $V(b)$ on $(\mathscr{A} \otimes E)/p$

$$V(b)\left(\sum b_j \dot{\otimes} y_j\right) = \sum bb_j \dot{\otimes} y_j .$$

By (4) $V(b)$ is well-defined and can be extended to $\mathscr{L}(\tilde{E})$. Also,

$$\| V(b) \| \leqslant \| b \| .$$

We now show that $(\tilde{E}, i, P, V)$ is a minimal dilation of U satisfying conditions (1.1.1) through (1.1.5). First, (assuming that $\tilde{\omega}$ has been extended to a mapping of $\tilde{E}$ into the completion of E) note that, for $x \in E$,

$$\tilde{\omega} i(x) = \lim_{\lambda \in \Lambda} U(u_\lambda)\, x = x$$

so that i is injective and $i^{-1} = \tilde{\omega} \mid i(E)$ which is continuous by (3). A simple calculation now shows that P is a projection of $\tilde{E}$ into the closure of $i(E)$. Writing $Pi = i\tilde{\omega} i = i$ one sees that P is the identity on $i(E)$ and, therefore, by continuity, on the closure of $i(E)$. V is clearly a continuous representation of $\mathscr{A}$ into $\mathscr{L}(\tilde{E})$ with $\| V \| \leqslant 1$. Since $PV(a)\, i = i\tilde{\omega} V(a)\, i = i\tilde{\omega}_a i = iU(a)$, $(\tilde{E}, i, P, V)$ is a dilation of U; $(\tilde{E}, i, P, V)$ is minimal since the vector space spanned by $\{V(a)\, i(x) : a \in \mathscr{A}, x \in E\}$ is $(\mathscr{A} \otimes E)/p$ which is clearly dense in $\tilde{E}$.

PROOF OF (1.1.1): For $b \in \mathscr{A}$, $x \in E$,

$$\lim_{\lambda \in \Lambda} \| V(u_\lambda)\, (b \dot{\otimes} x) - b \dot{\otimes} x \| = \lim_{\lambda \in \Lambda} \sup_{\|a\| \leqslant 1} \| U(au_\lambda b - ab)\, x \| = 0,$$

and therefore $\lim_{\lambda\in\Lambda}(V(u_\lambda)\, y = y$ for $y \in (\mathscr{A} \otimes E)/p$. To complete the proof, use the density of $(\mathscr{A} \otimes E)/p$ in $\tilde{E}$.

PROOF OF (1.1.2): If $U(ab) = U(a)\, U(b)$ for every $a \in \mathscr{A}$, then, for any $x \in E$,

$$V(b)\, ix - iU(b)\, x \,\| = \lim_{\lambda\in\Lambda} \sup_{\|a\|\leqslant 1} \| \, U(abu_\lambda)\, x - U(au_\lambda)\, U(b)\, x \, \|$$

$$= \lim_{\lambda\in\Lambda} \| \, U(abu_\lambda)\, x - U(au_\lambda b)\, x \, \| = 0.$$

Therefore $V(b)\, i = iU(b)$. The converse is clear.

PROOF OF (1.1.4): Using (1) and the fact that $\| V \| \leqslant 1$,

$$\| V(a)\, (b \,\dot{\otimes}\, x) \| \leqslant \| a \| \,\sigma \sup_{\|a\|\leqslant 1, \|c\|\leqslant 1} \| \, U(abc)\, x \, \| \qquad \text{for} \qquad x \in E,\, a, b \in \mathscr{A}.$$

Therefore $\lim_{a,\mathscr{F}} V(a)\, y = 0$ for $y \in (\mathscr{A} \otimes E)/p$. The rest follows from the fact that $\mathscr{F}$ is a bounded filter base and that $(\mathscr{A} \otimes E)/p$ is dense in $\tilde{E}$.

PROOF OF (1.1.3): If $U(abc) = 0$ for every $a \in \mathscr{A}$, $c \in \mathscr{A}$, then, by (1.1.4), $V(b)\, x = \lim_{d,\mathscr{F}} V(d)\, x = 0$ where $\mathscr{F} = \{\{b\}\}$.

REMARK. The proof of Theorem 1.1 shows that any dilation satisfying (1.1.4) satisfies (1.1.3).

THEOREM 1.2. *Let E be a normed space, $\mathscr{A}$ a Michael algebra, and U a continuous linear mapping of $\mathscr{A}$ into $\mathscr{L}(E)$. Let $(u_\lambda)_{\lambda\in\Lambda}$ be a bounded net in $\mathscr{A}$ and suppose that there exists a continuous semi-norm p on $\mathscr{A}$ such that, for any $x \in E$, $b \in \mathscr{A}$,*

$$\lim_{\lambda\in\Lambda} \sup_{p(a)\leqslant 1} \| \, U(au_\lambda - a)\, x \, \| = \lim_{\lambda\in\Lambda} \sup_{p(a)\leqslant 1} \| \, U(au_\lambda b - ab)\, x \, \| = 0,$$

$$\lim_{\lambda\in\Lambda} U(u_\lambda)\, x = x, \qquad \textit{and} \qquad \lim_{\lambda\in\Lambda} U(u_\lambda b)\, x = U(b)\, x.$$

Then there exists a minimal dilation of U having properties (1.1.1), (1.1.2), *and* (1.1.3) *of Theorem 1.1.*

PROOF. There exists a continuous multiplicative semi-norm $q \geqslant p$ on $\mathscr{A}$ and a number $M > 0$ such that $\| \, U(a) \, \| \leqslant Mq(a)$ for every $a \in \mathscr{A}$.

Letting ψ be the canonical homomorphism of $\mathscr{A}$ onto $\mathscr{A}/q$, there is a unique continuous linear mapping W from $\mathscr{A}/q$ into $\mathscr{L}(E)$ such that $U = W\psi$.

It is easy to see that W satisfies the hypotheses of Theorem 1.1 (with u_λ replaced by $\psi(u_\lambda)$). Therefore, by Theorem 1.1, there exists a minimal dilation $(\tilde{E}, i, P, V)$ of W satisfying (1.1.1), (1.1.2), and (1.1.3). Simple calculation now shows that $(\tilde{E}, i, P, V\psi)$ is a minimal dilation of U satisfying these three conditions.

REMARKS. (i) Theorem 1.1 was discovered by C. Ionescu Tulcea [5]. The proof presented here was suggested by the proof of Theorem 1 in [16]. (ii) The dilation constructed in the proof of Theorem 1.2 has the following property: if $\mathscr{F}$ is a bounded filter base on $\mathscr{A}$ such that

$$\lim_{b,\mathscr{F}} \sup_{p(a)\leqslant 1} \| U(abc)\, x \| = 0$$

for every $x \in E$, $b \in \mathscr{A}$, then $\lim_{\mathscr{F}} V = 0$ strongly in $\mathscr{L}(\tilde{E})$.

PROPOSITION 1.1. *Let $\mathscr{A}$ be a commutative algebra endowed with a topology, E a normed space, and U a continuous linear mapping of $\mathscr{A}$ into $\mathscr{L}(E)$. Suppose that U has the following property: if $b \in \mathscr{A}$ is such that $U(ab) = 0$ for every $a \in \mathscr{A}$, then $U(b) = 0$. (In particular, U has this property if there exists a net $(u_\lambda)_{\lambda\in\Lambda}$ in $\mathscr{A}$ such that $\lim_{\lambda\in\Lambda} U(u_\lambda b) = U(b)$ weakly for every $b \in \mathscr{A}$.) Then, every minimal dilation of U satisfying* (1.1.2) *satisfies* (1.1.3).

PROOF. Suppose that $(\tilde{E}, i, P, V)$ is a minimal dilation of U having property (1.1.2). Let $b \in \mathscr{A}$ such that $U(ab) = 0$ for every $a \in \mathscr{A}$. Then $U(b) = 0$ so that $U(ab) = 0 = U(a)\, U(b)$ for every $a \in \mathscr{A}$. Hence, by (1.1.2), $V(b)\, i = iU(b) = 0$. One then concludes that $V(b)\, V(a)\, i = V(a)\, V(b)\, i = 0$ for every $a \in \mathscr{A}$; i.e., $V(b) = 0$ on $\{V(a)\, ix = a \in \mathscr{A}, x \in E\}$. Thus, by minimality, $V(b) = 0$.

Whether Proposition 1.1 is valid if $\mathscr{A}$ is not commutative is not known. The converse of Proposition 1.1 is false as the following example shows.

EXAMPLE. Let $\tilde{E} = C_\infty(N^*)$ and let E be the set of all $x \in \tilde{E}$ such that $x(2n + 1) = x(2n + 2)$ for all $n \in N$. Define a projection P of $\tilde{E}$ onto E by the equations

$$(Px)\,(2n + 1) = (Px)\,(2n + 2) = x(2n + 1)$$

for $n \in N$. Let $\mathscr{A} = C^\infty(N^*)$, and, for $a \in \mathscr{A}$, $x \in \tilde{E}$, $V(a)\, x(n) = a(n)\, x(n)$ if n is odd and $V(a)\, x(n) = a(n + 1)\, x(n)$ if n is even. Then V is a continuous representation of $\mathscr{A}$ into $\mathscr{L}(\tilde{E})$. If we define i to be the injection of E into $\tilde{E}$ and $U(a) = PV(a) \mid E$ for $a \in \mathscr{A}$, then $(\tilde{E}, i, P, V)$ is a minimal dilation of U satisfying (1.1.3) but not (1.1.2). $(\tilde{E}, i, P, V)$ is minimal since the vector space spanned by $\{V(a)\,x : a \in \mathscr{A}, x \in E\}$ contains $\mathscr{K}(N^*)$. To verify that the dilation does not satisfy (1.1.2) it is enough to observe that U is a representation, but that there exist $a \in \mathscr{A}$ such that $U(a) \neq V(a) \mid E$; to verify that the dilation satisfies (1.1.2) one need only observe that $U(a) = 0$ if and only if $a(n) = 0$ for all odd $n \in N^*$.

2. Generalized Subscalar Operators

In this section we define and characterize generalized subscalar operators. We begin with some definitions, most of which are taken from [13].

A commutative algebra $\mathscr{A}$ together with a family $(\mathscr{A}_K)_{K \in \mathfrak{K}}$ of ideals of $\mathscr{A}$ and, for each $K \in \mathfrak{K}$, a bilinear map $(\varphi, a) \to \varphi \times_K a$ of $\mathscr{H}(K) \times \mathscr{A}_K$ into $\mathscr{A}_K$ is called a *distributional system* if

(1) $\mathscr{A}_\phi = \{0\}$, $\mathscr{A}_{K \cap L} = \mathscr{A}_K \cap \mathscr{A}_L$ for every $K \in \mathfrak{K}, L \in \mathfrak{K}$;

(2) if $K \in \mathfrak{K}$, $L \in \mathfrak{K}$, and $K \subset L$, then $\varphi \times_K a = \varphi \times_L a$ for every $\varphi \in \mathscr{H}(L)$, $a \in \mathscr{A}_K$;

(3) for any $K \in \mathfrak{K}$, $a \in \mathscr{A}_K$, $b \in \mathscr{A}_K$, $\varphi \in \mathscr{H}(K)$, $\psi \in \mathscr{H}(K)$, $(\varphi\psi) \times_K a = \varphi \times_K (\psi \times_K a)$, $\varphi \times_K (ab) = (\varphi \times_K a)\, b$, and $1 \times_K a = a$.

The subscript K on $\times_K$ will be omitted since, by (2), no ambiguity can result.

For a distributional system $\mathscr{A}$ and $K \in \mathfrak{K}$, an element $u \in \mathscr{A}$ is a *K-unit* if $ua = a$ for every $a \in \mathscr{A}_K$. The set of K-units of $\mathscr{A}$ will be denoted by $\mathscr{U}_K$. For $S \subset R^2$ define

$$\mathscr{A}_S = \bigcup_{K \in \mathfrak{K}(S)} \mathscr{A}_K, \qquad \mathscr{U}_S = \bigcap_{K \in \mathfrak{K}(S)} \mathscr{U}_K;$$

write also $\mathscr{A}_c = \mathscr{A}_C$. An element $u \in \mathscr{A}$ is *one over* $S \subset R^2$ if $u \in \mathscr{U}_Q$ for some neighborhood Q of S. A distributional system $\mathscr{A}$ is called *separating* if, for any $K \in \mathfrak{K}$ and any neighborhood Q of K, $\mathscr{U}_K \cap \mathscr{A}_Q \neq \phi$; $\mathscr{A}$ is *modular* if, for any closed subset F of R^2 and any $u \in \mathscr{A}$ one over F, $a - ua \in \mathscr{A}_{\complement F}$ for every $a \in \mathscr{A}_c$.

Let E be a Banach space, $\mathscr{A}$ a distributional system, and U a linear mapping of $\mathscr{A}$ into $\mathscr{L}(E)$. A net $(u_\lambda)_{\lambda \in \Lambda}$ of elements of $\mathscr{A}_c$ is an *approximate identity* for U if the net $(U(u_\lambda))_{\lambda \in \Lambda}$ converges simply to I in $\mathscr{L}(E)$ and, for any $a \in \mathscr{A}$, the net $(U(au_\lambda))_{\lambda \in \Lambda}$ converges simply to $U(a)$. U is an *$\mathscr{A}$-spectral mapping* if

(1) there exists an approximate identity for U, and (2) for every $K \in \mathfrak{K}$ $a \in \mathscr{A}_K$, the mapping $\varphi \to U(\varphi \times a)$ of $\mathscr{H}(K)$ into $\mathscr{L}(E)$ is continuous; U is an *$\mathscr{A}$-scalar mapping* for $T \in \mathscr{L}(E)$ if, in addition, $U(z \times a) = U(a)\,T$ for every $a \in \mathscr{A}_c$. An $\mathscr{A}$-spectral mapping which is also a representation is called an *$\mathscr{A}$-spectral representation.*

If U is a linear mapping of a distributional system $\mathscr{A}$ into $\mathscr{L}(E)$ and F is a closed subset of R^2, one says that U is *supported by* F if $U(u) = I$ and $U(ua) = U(a)$ for every $a \in \mathscr{A}$ and every $u \in \mathscr{A}$ which is one over F. By the same method used to prove Proposition 2.1, (1) of [13] one proves

PROPOSITION 2.1. *Let $\mathscr{A}$ be a modular distributional system, U an $\mathscr{A}$-spectral mapping, and F a closed subset of R^2. If $U(a) = 0$ for every $a \in \mathscr{A}_{\complement F}$, then U is supported by F.*

DEFINITION. Let $\mathscr{A}$ be a distributional system, E a Banach space, and $T \in \mathscr{L}(E)$. A linear function m of $\mathscr{A}$ into E is called a *mapping associated with* T if (1) for every $K \in \mathfrak{K}$, $a \in \mathscr{A}_K$, the mapping $\varphi \to m(\varphi \times a)$ of $\mathscr{H}(K)$ into E is continuous, and (2) for every $a \in \mathscr{A}_c$, $m(z \times a) = Tm(a)$.

REMARKS. 1. An $\mathscr{A}$-spectral mapping U is an $\mathscr{A}$-scalar mapping for T if and only if, for every $x' \in E'$, the function $a \to {}^tU(a)\,x'$ is a mapping associated with ${}^tT$. 2. This concept was first introduced by E. Bishop [1].

PROPOSITION 2.2. *If m is a mapping associated with T, then $m(a) = 0$ for every $a \in \mathscr{A}_{\complement \mathrm{sp}(T)}$.*

PROOF. Supposing that $a \in \mathscr{A}_K$ with $K \cap \mathrm{sp}(T) = \phi$, define $r : C \to E$ by $r(\lambda) = (\lambda - T)^{-1}\, m(a)$ if $\lambda \in \rho(T)$ and $r(\lambda) = m(\psi_\lambda \times a)$ if $\lambda \in \complement K$. To see that r is well-defined, write

$$m(\psi_\lambda \times a) = (\lambda - T)^{-1}\,(\lambda - T)\, m(\psi_\lambda \times a)$$
$$= (\lambda - T)^{-1}\, m((\lambda - z) \times (\psi_\lambda \times a)) = (\lambda - T)^{-1}\, m(a)$$

for $\lambda \in \rho(T) \cap \complement K$. Since $\lambda \to \psi_\lambda$ is an analytic function of $\complement$K into $\mathscr{H}(K)$, r is an entire function vanishing at ∞. Thus $r = 0$, and therefore $m(a) = 0$.

COROLLARY. *Let $\mathscr{A}$ be a modular separating distributional system, E a Banach space, and U an $\mathscr{A}$-scalar mapping for $T \in \mathscr{L}(E)$. Then U is supported by* $\mathrm{sp}(T)$.

PROOF. By Proposition 2.2 and Remark 1 above, ${}^tU(a) = 0$ for every $a \in \mathscr{A}_{\complement \mathrm{sp}(T)}$. The result now follows easily.

As a consequence of this Corollary, if $u \in \mathscr{A}_c$ is one over $\mathrm{sp}(T)$,

$T = U(z \times u)$; if, in addition, U is a representation, $TU(a) = U(a)\, T$ for every $a \in \mathscr{A}$.

Let E be a Banach space and $\mathscr{A}$ a distributional system. $T \in \mathscr{L}(E)$ is called an *$\mathscr{A}$-scalar operator* if there exists an $\mathscr{A}$-scalar representation for T. T is *$\mathscr{A}$-subscalar* if there exists a Banach space $\tilde{E}$, a linear bicontinuous injection i of E into $\tilde{E}$, a continuous projection P of $\tilde{E}$ onto $i(E)$ and a $\mathscr{A}$-scalar operator $\tilde{T} \in \mathscr{L}(\tilde{E})$ such that $\tilde{T}i = iT$. The object $(\tilde{E}, i, P, \tilde{T})$ is called an *$\mathscr{A}$-scalar extension* of T. If no confusion is possible, we will refer "by abuse of language" to "the $\mathscr{A}$-scalar extension $\tilde{T}$" rather than to "the $\mathscr{A}$-scalar extension $(\tilde{E}, i, P, \tilde{T})$."

A topology τ on a distributional system $\mathscr{A}$ is called *admissable* if, for any $K \in \mathfrak{K}$, $a \in \mathscr{A}_K$, the mapping $\varphi \to \varphi \times a$ of $\mathscr{K}(K)$ into $\mathscr{A}$ is continuous. The *final topology*, τ_f, on $\mathscr{A}$ is the finest admissable topology on $\mathscr{A}$; the *final Michael topology*, τ_m, on $\mathscr{A}$ is defined to be $m(\tau_f)$ (see Introduction).

Let τ be an admissable topology on a distributional system $\mathscr{A}$ and E a Banach space. A linear function U of $\mathscr{A}$ into $\mathscr{L}(E)$ is a *$(\mathscr{A}, \tau)$-spectral mapping* if (1) there exists an approximate identity for U, and (2) U is τ-continuous. One then defines *$(\mathscr{A}, \tau)$-spectral representation*, *$(\mathscr{A}, \tau)$-scalar mapping* [*representation*] for $T \in \mathscr{L}(E)$, *$(\mathscr{A}, \tau)$-scalar operator, and $(\mathscr{A}, \tau)$-subscalar operator* in the natural way.

REMARK. Using the Corollary to Proposition 4.1 and Corollary 2 to Proposition 4.2 in [13] one sees that, for $T \in \mathscr{L}(E)$, the following assertions are equivalent: (1) T is an $\mathscr{A}$-scalar [resp. $\mathscr{A}$-subscalar] operator; (2) T is a $(\mathscr{A}, \tau)$-scalar [resp. $(\mathscr{A}, \tau)$-subscalar] operator for some admissable topology τ on $\mathscr{A}$; (3) T is an $(\mathscr{A}, \tau_f)$-scalar [resp. $(\mathscr{A}, \tau_f)$-subscalar] operator; (4) T is an $(\mathscr{A}, \tau_m)$-scalar [resp. $(\mathscr{A}, \tau_m)$-subscalar] operator.

Suppose now that $\mathscr{A}$ is a distributional system endowed with an admissable topology τ, E is a Banach space, and $T \in \mathscr{L}(E)$ is an $(\mathscr{A}, \tau)$-subscalar operator. Let $\mathscr{E}(T; \mathscr{A}, \tau)$ be the class of all objects $(\tilde{E}, i, P, \tilde{T}, V)$ such that $(\tilde{E}, i, P, \tilde{T})$ is an $(\mathscr{A}, \tau)$-scalar extension of T and V is an $(\mathscr{A}, \tau)$-scalar representation for $\tilde{T}$. If $e_1 = (\tilde{E}_1, i_1, P_1, \tilde{T}_1, V_1)$ and $e_2 = (\tilde{E}_2, i_2, P_2, \tilde{T}_2, V_2)$ are elements of $\mathscr{E}(T; \mathscr{A}, \tau)$ write $e_1 \leqslant e_2$ if there exists a bicontinuous linear injection j of $\tilde{E}_1$ into $\tilde{E}_2$ such that

$$\begin{aligned} &i_2 = j i_1, \qquad jP_1 = P_2 j \\ &jV_1(a) = V_2(a)\, j \qquad \text{for every} \qquad a \in \mathscr{A}. \end{aligned} \tag{6}$$

The last of these equations implies that $j\tilde{T}_1 = \tilde{T}_2 j$. One says that e_1 and e_2 are *equivalent* if there exists an isomorphism (in the category of topological vector spaces) j of $\tilde{E}_1$ onto $\tilde{E}_2$ such that Eqs. (6) are satisfied. An element

$e \in \mathscr{E}(T; \mathscr{A}, \tau)$ is *minimal* if $e_1 \in \mathscr{E}(T; \mathscr{A}, \tau)$ and $e_1 \leqslant e$ imply e_1 is equivalent to e.

PROPOSITION 2.3. *Let $T \in \mathscr{L}(E)$ be an $(\mathscr{A}, \tau)$-subscalar operator and let $e = (\tilde{E}, i, P, \tilde{T}, V)$ be an element of $\mathscr{E}(T; \mathscr{A}, \tau)$. Then e is minimal in $\mathscr{E}(T; \mathscr{A}, \tau)$ if and only if $\{V(a)\, ix : a \in \mathscr{A},\ x \in E\}$ is total in $\tilde{E}$.*

PROOF. Supposing e minimal, let $\tilde{E}_1$ be the closed subspace of $\tilde{E}$ spanned by $\{V(a)\, ix : a \in \mathscr{A},\ x \in E\}$. Since $\tilde{E}_1 \supset i(E)$ and is invariant under each $V(a)$ $(a \in \mathscr{A})$, $e_1 = (\tilde{E}_1, i, P \mid \tilde{E}_1, \tilde{T} \mid \tilde{E}_1, W)$ $(W(a) = V(a) \mid \tilde{E}_1)$ is an element of $\mathscr{E}(T; \mathscr{A}, \tau)$ and $e_1 \leqslant e$. Consequently, e_1 is equivalent to e; this implies that $\tilde{E} = \tilde{E}_1$.

To prove the converse assertion suppose $\{V(a)\, ix : a \in \mathscr{A}, x \in E\}$ total in $\tilde{E}$. Let $e_1 \in \mathscr{E}(T; \mathscr{A}, \tau)$ with $e_1 \leqslant e$ and $j : \tilde{E}_1 \to \tilde{E}$ be a bicontinuous linear injection satisfying Eqs. (6). For any $a \in \mathscr{A}$, $x \in E$, $V(a)\, ix = jV_1(a)\, i_1 x$ and therefore $j(\tilde{E}_1)$ contains the vector space spanned by $\{V(a)\, ix : a \in \mathscr{A}, x \in E\}$. One now concludes that j is an isomorphism; hence the result.

One calls an $(\mathscr{A}, \tau)$-scalar extension $(\tilde{E}_1, i, P, \tilde{T})$ of T *minimal* if there exists an $(\mathscr{A}, \tau)$-scalar representation V for $\tilde{T}$ such that $(\tilde{E}_1, i, P, \tilde{T}, V)$ is minimal in $\mathscr{E}(T; \mathscr{A}, \tau)$. By Proposition 2.3 the $(\mathscr{A}, \tau)$-scalar extension $\tilde{T}$ of T is minimal if and only if there exists an $(\mathscr{A}, \tau)$-scalar representation V for $\tilde{T}$ such that $\{V(a)\, ix : a \in \mathscr{A}, x \in E\}$ is total in $\tilde{E}$. Two consequences of this fact should be noted. (1) For any $(\mathscr{A}, \tau)$-subscalar operator, there exists a minimal $(\mathscr{A}, \tau)$-scalar extension. (2) The definition of minimality given here coincides with the definitions given in [5] and [14].

THEOREM 2.1. *Let $\mathscr{A}$ be a modular separating distributional system, τ an admissable topology on $\mathscr{A}$, E a Banach space, and $T \in \mathscr{L}(E)$. Then T is $(\mathscr{A}, \tau)$-subscalar if and only if there exists an $(\mathscr{A}, m(\tau))$-scalar mapping for T.*

PROOF. Supposing that T is $(\mathscr{A}, \tau)$-subscalar, let $(\tilde{E}, i, P, \tilde{T}, V)$ be an element of $\mathscr{E}(T; \mathscr{A}, \tau)$, and define $U(a) = i^{-1}PV(a)\, i$ for $a \in \mathscr{A}$. ($U(a)$ is well-defined since i is injective and P maps $\tilde{E}$ into $i(E)$.) U is clearly linear. Any approximate identity for V is an approximate identity for U. V is $m(\tau)$-continuous by Proposition 4.2 in [13] and therefore so is U. For any $a \in \mathscr{A}_c$,

$$U(z \times a) = i^{-1}PV(z \times a)\, i = i^{-1}PV(a)\, \tilde{T}i = i^{-1}PV(a)\, iT = U(a)\, T.$$

Conversely, let U be an $(\mathscr{A}, m(\tau))$-scalar mapping for T and let $u_0 \in \mathscr{A}_c$ be one over $\mathrm{sp}(T)$. By the Corollary to Proposition 2.2 $U(u_0) = I$ and $U(u_0 a) = U(a)$ for every $a \in \mathscr{A}$. Therefore, by Theorem 1.2 (take $\Lambda = \{0\}$) there exists a minimal dilation $(\tilde{E}, i, P, V)$ of U having properties (1.1.1), (1.1.2), and (1.1.3). If $b \in \mathscr{A}_{\complement \mathrm{sp}(T)}$, then $ab \in \mathscr{A}_{\complement \mathrm{sp}(T)}$ for every $a \in \mathscr{A}$; it follows from (1.1.3) that $V(b) = 0$. Thus V is supported by $\mathrm{sp}(T)$. By Proposition 4.2 in [13] V is an $(\mathscr{A}, \tau)$-spectral representation.

We now claim that, with $\tilde{T} = V(z \times u_0)$, $(\tilde{E}, i, P, \tilde{T})$ is a minimal $(\mathscr{A}, \tau)$-scalar extension of T. For any $a \in \mathscr{A}_c$,

$$V(z \times a) = V((z \times a)\, u_0) = V((z \times u_0)\, a) = V(a)\, \tilde{T}$$

so that V is an $(\mathscr{A}, \tau)$-scalar representation for $\tilde{T}$. For any $a \in \mathscr{A}$

$$\begin{aligned} U(a(z \times u_0)) &= U(z \times (au_0)) = U(au_0)\, T = U(a)\, T = U(a)\, U(u_0)\, T \\ &= U(a)\, U(z \times u_0). \end{aligned}$$

By (1.1.2), $\tilde{T}i = V(z \times u_0)\, i = iU(z \times u_0) = iT$. Thus we have shown that $(\tilde{E}, i, P, \tilde{T})$ is an $(\mathscr{A}, \tau)$-scalar extension of T. $(\tilde{E}, i, P, \tilde{T})$ is minimal by Proposition 2.3.

COROLLARY 1. *Let $\mathscr{A}$ be a modular separating distributional system, E a Banach space, and $T \in \mathscr{L}(E)$. Then T is $\mathscr{A}$-subscalar if and only if there exists an $(\mathscr{A}, \tau_m)$-scalar mapping for T.*

PROOF. By Theorem 2.1 and the remark after the definition of $(\mathscr{A}, \tau)$-subscalar operator, each of these assertions is equivalent to the assertion that T is $(\mathscr{A}, \tau_m)$-subscalar.

COROLLARY 2. *Let $\mathscr{A}$ be a modular separating distributional system endowed with an admissable topology τ, E a Banach space, and $T \in \mathscr{L}(E)$ an $(\mathscr{A}, \tau)$-subscalar operator. Then there exists a minimal $(\mathscr{A}, \tau)$-scalar extension $\tilde{T}$ of T such that* $\mathrm{sp}(\tilde{T}) \subset \mathrm{sp}(T)$.

PROOF. Let U be an $(\mathscr{A}, m(\tau))$-scalar mapping for T and let $(\tilde{E}, i, P, V)$ be a minimal dilation of U satisfying conditions (1.1.1), (1.1.2), and (1.1.3) of Theorem 1.1. As in the proof of Theorem 2.1, $\tilde{T} = V(z \times u_0)$ is a minimal $(\mathscr{A}, \tau)$-scalar extension of T, and V is supported by $\mathrm{sp}(T)$. By Proposition 3.1 of [13], $\mathrm{sp}(\tilde{T}) \subset \mathrm{sp}(T)$.

3. LIMIT THEOREMS

In this section we prove that, with certain restrictions, the strong limit of a net of generalized scalar operators is a generalized subscalar operator. The methods are analogs of those of Bishop [1].

Let $\mathscr{A}$ be a distributional system and N an absorbing subset of $\mathscr{A}$. For $a \in \mathscr{A}$ define

$$\| a \|_N = \inf \left\{ s > 0 : \left(\frac{a}{s} \right) \in N \right\}.$$

If m is a linear mapping of $\mathscr{A}$ into a Banach space, let

$$\| m \|_N = \sup_{a \in N} \| m(a) \| .$$

THEOREM 3.1. *Let $\mathscr{A}$ be a distributional system endowed with an admissable topology, E a Banach space, and $(T_\lambda)_{\lambda \in \Lambda}$ a net of linear operators on E converging strongly to $T \in \mathscr{L}(E)$. Suppose that, for each $\lambda \in \Lambda$, $m_\lambda : \mathscr{A} \to E'$ is a mapping associated with ${}^tT_\lambda$ and that $m \in \mathscr{L}(\mathscr{A}, E')$ is such that*

$$c = \sup_{\lambda \in \Lambda} \| m_\lambda \|_N < \infty \tag{3.1.1}$$

for some absorbing set $N \subset \mathscr{A}$, and for every $a \in \mathscr{A}_c$, $x \in E$,

$$\lim_{\lambda \in \Lambda} \langle x, m_\lambda(a) \rangle = \langle x, m(a) \rangle. \tag{3.1.2}$$

Then m is a mapping associated with ${}^tT$.

PROOF. For every $K \in \mathfrak{K}$, $a \in \mathscr{A}_K$, the mapping $\varphi \to m(\varphi \times a)$ of $\mathscr{H}(K)$ into E is continuous by the continuity of m. Now let $x \in E$, $a \in \mathscr{A}_c$, and $s > 0$ such that $(a/s) \in N$. Then

$$\begin{aligned} &| \langle x, {}^tTm(a) \rangle - \langle x, m(z \times a) \rangle | \\ = &| \langle Tx, m(a) \rangle - \langle x, m(z \times a) \rangle | \leqslant | \langle Tx, m(a) \rangle - \langle Tx, m_\lambda(a) \rangle | \\ &+ | \langle Tx, m_\lambda(a) \rangle - \langle T_\lambda x, m_\lambda(a) \rangle | + | \langle T_\lambda x, m_\lambda(a) \rangle - \langle x, m(z \times a) \rangle | . \end{aligned}$$

To complete the proof we need only show that each of the three terms on the right side of this inequality converge to 0. For the first and third terms use condition (3.1.2); in the third term note that

$$\langle T_\lambda x, m_\lambda(a) \rangle = \langle x, {}^tT_\lambda m_\lambda(a) \rangle = \langle x, m_\lambda(z \times a) \rangle.$$

For the second term write

$$\begin{aligned} | \langle Tx, m_\lambda(a) \rangle - \langle T_\lambda x, m_\lambda(a) \rangle | &\leqslant \| Tx - T_\lambda x \| \, \| m_\lambda(a) \| \\ &\leqslant cs \, \| Tx - T_\lambda x \| . \end{aligned}$$

We have therefore proved that ${}^tTm(a) = m(z \times a)$.

For a distributional system $\mathscr{A}$, an admissable topology τ on $\mathscr{A}$, an absorbing subset N of $\mathscr{A}$, a positive number c, and a Banach space E, we define $\mathscr{S}_{N,c}(\mathscr{A}, \tau; E)$ to be the set of $(\mathscr{A}, \tau)$-scalar operators $T \in \mathscr{L}(E)$ such that there exists an $(\mathscr{A}, \tau)$-scalar representation V for T with $\| V \|_N \leqslant c$. In the proof of the next theorem, we will use the topology on $\mathscr{L}(\mathscr{A}, \mathscr{L}(E'))$ generated by the semi-norms

$$U \to | \langle x, U(a)\, x' \rangle |$$

for $x \in E$, $a \in \mathscr{A}$, $x' \in E'$. In this topology $\{U \in \mathscr{L}(\mathscr{A}, \mathscr{L}(E')) : \| U \|_N \leqslant c\}$ is compact.

THEOREM. 3.2. *Suppose that $\mathscr{A}$ is a modular distributional system and that τ is an admissable vector space topology on $\mathscr{A}$ with the following property: there exists a sequence $(u_r)_{r \in N^*}$ of elements of $\mathscr{A}_c$ such that $u_r \in \mathscr{U}_{B(r)}$ for every $r \in N^*$ and $\{r(u_r - u_{r'}) \times \psi_0 : r \in N^*, r' \in N^*\}$ is bounded in the topology τ. If E is a Banach space and $T \in \mathscr{L}(E)$ is in the strong closure of $\mathscr{S}_{N,c}(\mathscr{A}, \tau; E)$ for some neighborhood N of 0 in $\mathscr{A}$ and some $c > 0$, then there exists an $m(\tau)$-continuous linear mapping U of $\mathscr{A}$ into $\mathscr{L}(E')$ having the following properties:*

(3.2.1) *for each $x' \in E'$, the mapping $a \to U(a)\, x'$ is a mapping associated with ${}^tT$;*

(3.2.2) *U is supported by* $\operatorname{sp}(T)$.

PROOF. Let $(T_\lambda)_{\lambda \in \Lambda}$ be a net in $\mathscr{S}_{N,c}(\mathscr{A}, \tau; E)$ converging strongly to T and, for each $\lambda \in \Lambda$, let V_λ be an $(\mathscr{A}, \tau)$-scalar representation for T such that $\| V_\lambda \|_N \leqslant c$. Suppose, for the remainder of the proof, that $\mathscr{A}$ is endowed with the topology $m(\tau)$. By Lemma 4.1 of [13] the mapping $a \to {}^tV_\lambda(a)$ from $\mathscr{A}$ into $\mathscr{L}(E')$ is $m(\tau)$-continuous.

Let U be a cluster point of $({}^tV_\lambda)_{\lambda \in \Lambda}$ in the topology on $\mathscr{L}(\mathscr{A}, \mathscr{L}(E'))$ described above; we can assume that U is the limit of the net $({}^tV_r)_{\lambda \in \Lambda}$. Property (3.2.1) follows from Theorem 3.1 and Remark 1 after the definition of mapping associated with T.

To prove property (3.2.2) let $(u_r)_{r \in N^*}$ be a sequence of elements of $\mathscr{A}_c$ as described in the statement of the theorem. There exists $s > 0$ such that $sr\psi_0 \times (u_r - u_{r'}) \in N$ for every $r \in N^*$, $r' \in N^*$, and therefore

$$sr \,\| V_\lambda(\psi_0 \times (u_r - u_{r'})) \| \leqslant c \qquad \text{for every} \qquad \lambda \in \Lambda.$$

Hence, for any $x \in E$, $r \in N^*$, $r' \in N^*$,

$$\begin{aligned} \| V_\lambda(u_r - u_{r'})\, x \| &= \| V_\lambda(z \times \psi_0 \times (u_r - u_{r'}))\, x \| \\ &= \| V_\lambda(\psi_0 \times (u_r - u_{r'}))\, T_\lambda x \| \\ &\leqslant \| V_\lambda(\psi_0 \times (u_r - u_{r'})) \| \, \| T_\lambda x \| \leqslant \frac{c}{sr} (\| Tx \| + 1) \end{aligned}$$

for "λ large."

Now let $x \in E$, $x' \in E'$, $\epsilon > 0$ be arbitrary. Choose $r \in N^*$ so that $r \geqslant 1/\epsilon$ and $\operatorname{sp}(T)$ is contained in the interior of $B(r)$. Let $\lambda \in \Lambda$ such that

$$| \langle x, {}^tV_\lambda(u_r)\, x' - U(u_r)\, x' \rangle | \leqslant \epsilon$$

and

$$\| V_\lambda(u_r - u_{r'})\, x \| \leqslant \frac{c}{sr} (\| Tx \| + 1) \qquad \text{for all} \qquad r' \in N^*.$$

Then choose $r' \in N^*$ so that $V_\lambda(u_{r'}) = I$ (for example, so that $\mathrm{sp}(T_\lambda)$ is contained in the interior of $B(r')$). One concludes that

$$
\begin{aligned}
|\langle x, U(u_r)\,x' - x'\rangle| &\leqslant |\langle x, U(u_r)\,x' - {}^tV_\lambda(u_r)\,x'\rangle| \\
&\quad + |\langle x, {}^tV_\lambda(u_r - u_{r'})\,x'\rangle| + |\langle x, {}^tV_\lambda(u_{r'})\,x' - x'\rangle| \\
&\leqslant \epsilon + \frac{c}{sr}(\|Tx\| + 1)\,\|x'\| \\
&\leqslant \left(1 + \frac{c}{s}(\|Tx\| + 1)\,\|x'\|\right)\epsilon.
\end{aligned}
$$

If $u \in \mathscr{A}$ is one over $\mathrm{sp}(T)$, $U(u) = U(u_r)$ by (3.2.1), Proposition 2.2, and the modularity of $\mathscr{A}$. From the above calculations one deduces that $U(u) = I$.

The proof of the fact that $U(ua) = U(a)$ for every $u \in \mathscr{A}$ one over $\mathrm{sp}(T)$ is similar to the above. One uses the inequality

$$
\begin{aligned}
|\langle x, U(u_r a)\,x' - U(a)\,x'\rangle| &\leqslant |\langle x, U(u_r a)\,x' - {}^tV_\lambda(u_r a)\,x'\rangle| \\
&\quad + |\langle x, {}^tV_\lambda(u_r - u_{r'})\,{}^tV_\lambda(a)\,x'\rangle| + |\langle x, {}^tV_\lambda(a)\,x' - U(a)\,x'\rangle|.
\end{aligned}
$$

COROLLARY [7]. *Let E be a reflexive Banach space, and let $(\mathscr{A}, \tau)$ be as in Theorem 3.2. If $T \in \mathscr{L}(E)$ is in the strong closure of $\mathscr{S}_{N,c}(\mathscr{A}, \tau; E)$ for some neighborhood N of 0 in $\mathscr{A}$ and some $c > 0$, then T is $(\mathscr{A}, \tau)$-subscalar.*

PROOF. Let U be as in the conclusion of Theorem 3.2. The function $a \to \nu^{-1}\,{}^tU(a)\,\nu$ (ν the canonical isomorphism of E onto E'') is an $(\mathscr{A}, m(\tau))$-scalar mapping for T. The result now follows from Theorem 2.1.

REMARKS. (i) The mapping U constructed in the proof of Theorem 3.2 is supported by the closure of $\bigcup_{\lambda\in\Lambda} \mathrm{sp}(T_\lambda)$. (ii) Many distributional systems have the property described in Theorem 3.2. In particular the examples listed on p. 179 of [9] have this property. If $\mathscr{A} = \mathscr{A}_c$ and $\mathscr{A}$ has a unit 1, then $\mathscr{A}$ has this property (take $u_r = 1$). Consequently Examples 3 through 6 of [13], as well as Example 2 when S is bounded, have this property.

We now procede to prove that, for some of the distributional systems mentioned above, every $\mathscr{A}$-subscalar operator with "small spectrum" is $\mathscr{A}$-scalar.

PROPOSITION 3.1. *Let $\mathscr{A}$ and $\mathscr{A}'$ be semi-topological algebras, and let $U \in \mathscr{L}(\mathscr{A}, \mathscr{A}')$. Suppose there exists a total set $T \subset \mathscr{A}$ such that $U(st) = U(s)\,U(t)$ for every $s \in T$, $t \in T$. Then U is a representation.*

The proof of this proposition will be omitted since it is trivial.

Corollary. *Let $\mathscr{A}$ be a separating modular distributional system endowed with an admissable topology τ. Suppose $\mathscr{A}$ has a unit $1 \in \mathscr{A}_K (K \in \mathfrak{K})$ and that*

$$\{((\mu - z)^n \psi_\lambda^m) \times 1 : \mu \in C, \lambda \in \complement K, n \in N, m \in N\}$$

is total in $(\mathscr{A}, m(\tau))$. Then every $(\mathscr{A}, \tau)$-subscalar operator with spectrum contained in K is $(\mathscr{A}, \tau)$-scalar. If

$$\{(\mu - z)^n \times 1 : \mu \in C, n \in N\}$$

is total in $(\mathscr{A}, m(\tau))$, then every $(\mathscr{A}, \tau)$-subscalar operator is $(\mathscr{A}, \tau)$-scalar.

Proof. Let $T \in \mathscr{L}(E)$ be $(\mathscr{A}, \tau)$-subscalar, and let U be an $(\mathscr{A}, m(\tau))$-scalar mapping for T. For any $a \in \mathscr{A}$, $U(z \times a) = U(a)\,T$ so that, $U(z \times 1) = T$. One concludes by induction that

$$\begin{aligned} U(a((\mu - z)^n \times 1)) = U((\mu - z)^n \times a) &= U(a)\,(\mu - T)^n \\ &= U(a)\,U((\mu - z)^n \times 1) \end{aligned}$$

for any $a \in \mathscr{A}$, $\mu \in C$, $n \in N$. From the fact that $a = \psi_\lambda^n \times ((\lambda - z)^n \times a)$ one concludes

$$U(a(\psi_\lambda^n \times 1)) = U(a)\,(\lambda - T)^{-n} = U(a)\,U(\psi_\lambda^n \times 1)$$

for $a \in \mathscr{A}$, $\lambda \in \complement K$, $n \in N$. By Proposition 3.1 U is a representation; since $m(\tau) \subset \tau$, U is τ-continuous, and therefore T is $(\mathscr{A}, \tau)$-scalar.

Examples. 1. Let Δ be a compact interval in R and $n \in N \cup \{\infty\}$. By [11, Lemma 2.1] every $C^n(\Delta)$-subscalar operator is $C^n(\Delta)$-scalar. The result also holds if Δ is replaced by R.

2. Let γ be a C^n-curve [11] ($n \in N \cup \{\infty\}$). By [11, Lemma 1.1] every $C^n(\gamma)$-subscalar operator with spectrum contained in $\gamma(T^1)$ is $C^n(\gamma)$-scalar. Note that the assumption on the spectrum is essential.

By combining the Corollary to Theorem 3.2 with the above examples one proves

Proposition 3.2. *Let E be a reflexive Banach space, $T \in \mathscr{L}(E)$ $n \in N \cup \{\infty\}$, and $c > 0$. (1) If T is in the strong closure of $\mathscr{S}_{N,c}(C^n(R); E)$ for some neighborhood N of 0 in $C^n(R)$, then T is $C^n(R)$-scalar. (2) If γ is a C^n-curve, T is in the strong closure of $\mathscr{S}_{N,c}(C^n(\gamma); E)$ for some neighborhood N of 0 in $C^n(\gamma)$, and* $\operatorname{sp}(T) \subset \gamma(T^1)$, *then T is $C^n(\gamma)$-scalar.*

References

1. E. Bishop. Spectral theory for operators on a Banach space. *Trans. Amer. Math. Soc.* **86** (1957), 414–445.

2. H. Dowson. Restrictions of spectral operators. *Proc. London Math. Soc.* **XV** (1965), 437–457.
3. C. Foias. Une application des distributions vectorielles a la theorie spectrale, *Bull. Sci. Math.* (2) **84** (1960), 147–158.
4. P. Halmos. Normal dilations and extensions of operators, *Summa Brasil. Math.* **2** (1950), 125–134.
5. C. Ionescu Tulcea. Scalar dilations and scalar extensions of operators on Banach spaces (I). *J. Math. Mech.* **14** (1965), 841–865.
6. C. Ionescu Tulcea. "Notes on Spectral Theory" (unpublished).
7. C. Ionescu Tulcea and S. Plafker. Dilatations et extensions scalaires sur les espaces de Banach. *C. R. Acad. Sci. Paris* **265** (1967), 734–735.
8. S. Kantorovitz. Classification of operators by means of their operational calculus. *Trans. Amer. Math. Soc.* **115** (1965), 194–224.
9. F–Y. Maeda. Generalized spectral operators on locally convex spaces. *Pacific J. Math.* **13** (1963), 177–192.
10. F–Y. Maeda. Generalized unitary operators. *Bull. Amer. Math. Soc.* **71** (1965), 631–633.
11. F–Y. Maeda. Generalized scalar operators whose spectra are contained in a Jordan curve. *Ill. J. Math.* **10** (1966), 431–459.
12. E. Michael. Locally multiplicatively-convex topological algebras. *Mem. Amer. Math. Soc.* **11** (1952).
13. S. Plafker. Spectral representations for a general class of operators on a locally convex space. *Ill. J. Math.* (to appear).
14. W. Saffern. "Subscalar operators," Ph. D. Thesis, Columbia University, 1962.
15. J. Simpson. On spectral measures and spectral operators, Ph. D. Thesis, Yale University (1961).
16. W. Stinespring. Positive functions on C*-algebras, *Proc. Amer. Math. Soc.* **6** (1955), 211–216.
17. H. Umegaki. Positive definite function and direct product Hilbert space. *Tohoku Math. J.* **7** (1955), 206–211.
18. L. Waelbroek. Le calcul symbolique dans les algebras commutatives. *J. Math. Pures Appl.* **33** (1964), 147–186.